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Hitting of a line or a half-line in the plane by two-dimensional symmetric stable Lévy processes

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Abstract

Let $(X(t), Y(t))$ be a symmetric α-stable Lévy process on $\mathbb{R}^2$ with $1 < \alpha \leq 2$ and $L_Y(t)$ be the local time at 0 for $Y(t)$. A multivariate asymptotic estimate is obtained involving the first hitting time and place of the positive half of the X-axis, and $L_Y(\cdot)$ up to then. The method is based on the fluctuation identities for two-dimensional processes and the same method is applicable for a wider class of processes.

When $(X(0), Y(0)) = (0, 1)$, the law of the first hitting place of the whole X-axis is shown to have the explicit density const$/\Psi(1, x)$ where Ψ is the characteristic exponent.

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1. Introduction and the result

Let $S(N) = (S_1(N), S_2(N))$ be a random walk on $\mathbb{Z}^2$ starting from the origin, V_+ be the non-negative half of the first coordinate axis:

$$V_+ = \left\{(n, 0) \in \mathbb{Z}^2 | n \geq 0\right\}$$

and $\tau_{V+} = \inf\{N \geq 1 | S(N) \in V_+\}$. Lawler [16] obtained the following estimate.

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Theorem A (*Lawler [16]*). *When $S(N)$ is the simple random walk on $\mathbb{Z}^2$,*

$$P[\tau_{V+} > k] \asymp k^{-1/4} \quad \text{as } k \to \infty,$$

where $\asymp$ means that the ratio between both sides remains bounded, away from both 0 and ∞.

This result is applied to estimates for the Hausdorff dimension of the clusters in the diffusion limited aggregation(DLA) model. Fukai [8] generalized Lawler's result and showed the following.

Theorem B (*Fukai [8]*). *Assume $(S_1(N), S_2(N))$ and $(-S_1(N), S_2(N))$ have the same law, $E[S_2(1)] = 0$, $E[S_2(1)^2] < \infty$, and some technical conditions hold. Then it holds*

$$P[\tau_{V+} > k] \sim C_0 k^{-1/4} \quad \text{as } k \to +\infty$$

where $\sim$ means that the ratio between both sides converges to 1. Moreover, $C_0 = \Gamma(3/4)^{-1} (P[S(\tau_{V+}) \neq (0,0)])^{1/2}(2E[S_2(1)^2])^{1/4}$.

Fukai also expressed the constant C_0 in terms of the characteristic function $\phi(\theta_1, \theta_2) = E[\exp(\mathrm{i}\theta_1 S_1(1) + \mathrm{i}\theta_2 S_2(1))]$. When $S(N)$ is the simple random walk on $\mathbb{Z}^2$, we have $C_0 = 2^{-1}\Gamma(3/4)^{-1}\sqrt{1+\sqrt{2}}$. We refer the reader to Theorem D(iii) for an analogous formula for the Brownian motion on $\mathbb{R}^2$.

Let $L(0) = 0$ and $L(N) = \#\{1 \leq j \leq N | S_2(j) = 0\}$. We say that $S(N)$ is genuinely two-dimensional if, for any $N \geq 1$, $S(N)$ is not supported on any one-dimensional subgroup. In [12], the author studied a trivariate asymptotic estimate involving τ_{V+} as well as the first hitting place $S_1(\tau_{V+})$ and the sojourn time $L(\tau_{V+})$ on the first axis up to τ_{V+}.

Theorem C (*[12]*). *Assume that $S(N)$ is genuinely two-dimensional, centered, and square-integrable and satisfies either $\phi(\theta_1, \theta_2) = \phi(-\theta_1, \theta_2)$ or $\phi(\theta_1, \theta_2) = \phi(-\theta_1, -\theta_2)$. Then for any $\mu_0 \geq 0$, $\mu_1 \geq 0$, $\mu_2 \geq 0$ such that $\mu_0 + \mu_1 + \mu_2 > 0$, there exists $C_1(\mu_0, \mu_1, \mu_2, \phi) > 0$ such that*

$$1 - E[\mathrm{e}^{-\mu_0 s^2 L(\tau_{V+}) - \mu_1 s^2 S_1(\tau_{V+}) - \mu_2 s^4 \tau_{V+}}] \sim C_1(\mu_0, \mu_1, \mu_2, \phi) s, \tag{1.1}$$

as $s \to +0$. Moreover, we have

$$C_1(\mu_0, \mu_1, \mu_2, \phi) = C_3^{-1/4}\Gamma\left(\frac{3}{4}\right) C_0(\phi) \exp\left(I\left(\mu_0, \sqrt{C_2}\mu_1, \frac{C_3\mu_2}{2}\right)\right).$$

Here $I\left(\mu_0, \sqrt{C_2}\mu_1, \frac{C_3\mu_2}{2}\right)$, C_2 and C_3 are defined as follows. Let $q_{ij} = E[S_i(1)S_j(1)]$ for $i, j \in \{1, 2\}$, and let $2\pi/A_2$ be the smallest positive θ_2 such that $\phi(0, \theta_2) = 1$. We set $C_2 := (q_{11}q_{22} - q_{12}^2)/A_2^2 \equiv (\det \mathrm{Cov}(S, S))/A_2^2$, $C_3 := (2q_{22})/A_2^2$, and

$$I(\mu_0, \mu_1, \mu_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\log\left(\mu_0 + \sqrt{\mu_1^2 t^2 + 2\mu_2}\right)}{t^2 + 1} \mathrm{d}t.$$

Note that much more precise information can be retrieved from (1.1) than from formulas for individual variables, such as the tail probability concerning both the first hitting time and place.

In [11], the author studied the case for two-dimensional Brownian motion. Let $(B_1(t), B_2(t))$ be a standard Brownian motion on $\mathbb{R}^2$ starting from (x, y). Its probability and expectation are denoted by $P_{(x,y)}$ and $E_{(x,y)}$, respectively. Let $L_2(t)$ be the local time at 0 for $B_2(\cdot)$: $L_2(t) = \lim_{\varepsilon\to+0} \frac{1}{2\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(B_2(s))\mathrm{d}s$. For $a \in \mathbb{R}$, we set

$$\tau(a) = \inf\{t \geq 0 | B_2(t) = 0, B_1(t) \geq a\}.$$

Note that $\tau(0)$ is the first hitting time of the non-negative half of the first coordinate axis. Furthermore, $L_2(\tau(0))$ and $B_1(\tau(0))$ are the local time spent before $\tau(0)$ on the negative half of the first coordinate axis and the first hitting place on the positive half, respectively. To state the result in [11], we introduce a holomorphic function on the upper half plane. Let $\mathbb{C}_+ = \{z \in \mathbb{C} | \Im z > 0\}$, $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} | \Im z \geq 0\}$ and set

$$\varphi(z;\mu_0,\mu_2) = \exp\left(\frac{-1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{z}{t^2-z^2}\log(\mu_0+\sqrt{t^2+2\mu_2})\mathrm{d}t\right)$$

for $z \in \mathbb{C}_+$ and $\mu_i \geq 0$ $(i = 0, 2)$ such that $\mu_0 + \mu_2 > 0$. We can extend $\varphi(z;\mu_0,\mu_2)$ to $z \in \mathbb{R}$ by continuity. We also set

$$\varphi(z; 0, 0) = 1/\sqrt{-\mathrm{i}z}$$

for $z \in \overline{\mathbb{C}_+} \setminus \{0\}$, where we employ the branch of $\sqrt{\cdot}$ such that $\sqrt{1} = 1$.

Theorem D (*[11]*).

(i) *Let* $a > 0$, $\mu_i \geq 0$ $(i = 0, 1, 2)$, *and* $\mu_0 + \mu_1 + \mu_2 > 0$. *We have*

$$\begin{aligned}&E_{(-a,0)}\left[\mathrm{e}^{-\mu_0 L_2(\tau(0))-\mu_1 B_1(\tau(0))-\mu_2\tau(0)}\right]\\ &\quad= \mathrm{e}^{\mu_1 a} - \frac{\mathrm{e}^{\mu_1 a}}{\varphi(\mathrm{i}\mu_1;\mu_0,\mu_2)}\int_{-\infty}^{\infty}\mathrm{d}\theta\frac{1-\mathrm{e}^{-(\mu_1+\mathrm{i}\theta)a}}{2\pi(\mu_1+\mathrm{i}\theta)}\varphi(\theta;\mu_0,\mu_2).\end{aligned}$$

(ii) *It holds, as* $s \to +0$,

$$1 - E_{(-a,0)}\left[\mathrm{e}^{-\mu_0 s^2 L_2(\tau(0))-\mu_1 s^2 B_1(\tau(0))-\mu_2 s^4\tau(0)}\right] \sim \frac{2\sqrt{a}}{\sqrt{\pi}}\exp(I(\mu_0,\mu_1,\mu_2))s.$$

(iii) *It holds, as* $T \to +\infty$, $P_{(-a,0)}[\tau(0) > T] \sim \frac{2^{5/4}\sqrt{a}}{\sqrt{\pi}\Gamma(3/4)}T^{-1/4}$.

See [11] for the result for other starting points.

Remark 1.1. We compare Theorem D with other literature ([15], Theorem C, [14,13]).

(1) Kulczycki et al. [15,11] study the same object $L_2(\tau(0))$ and reveal its different aspects although we have not succeeded in verifying that they are in accordance with each other.

On one hand, [15, Corollary 2] obtains the density for $L_2(\tau(0))$:

$$P_{(-a,0)}[L_2(\tau(0)) \in \mathrm{d}t] = \frac{a}{\pi(a^2+t^2)}\exp\left(\frac{1}{\pi}\int_0^{\infty}\frac{\log\left(\frac{t}{x}+s\right)}{1+s^2}\mathrm{d}s\right)\mathrm{d}t \tag{1.2}$$

for $a > 0$ and $t > 0$. On the other hand, we put $\mu_1 = \mu_2 = 0$ in Theorem D(i) to obtain its Laplace transform:

$$E_{(-a,0)}\left[\mathrm{e}^{-\mu_0 L_2(\tau(0))}\right] = 1 - \sqrt{\mu_0}\int_{-\infty}^{\infty}\mathrm{d}\theta\frac{1-\mathrm{e}^{-\mathrm{i}\theta a}}{2\pi\mathrm{i}\theta}\varphi(\theta;\mu_0,0). \tag{1.3}$$

There is also a similarity in methods. Kulczycki et al. [15] regards a Cauchy process as the trace on a line of the two-dimensional Brownian motion, which is the subject of [11]. The method of [11] is based on the two-dimensional Lévy process $(B_1(L_2^{-1}(t)), L_2^{-1}(t))$ and $B_1(L_2^{-1}(t))$ is of course a Cauchy process. In the present paper we will also introduce a two-dimensional Lévy process in a similar manner.

(2) In view of Donsker's invariance principle, presence of $I(\mu_0, \mu_1, \mu_2)$ in both Theorems C and D(ii) seems to be a matter of course since properties of the tail probability should be persistent through a limiting procedure for the scaled random walk.

(3) Let $B(t)$ be the one-dimensional standard Brownian motion starting at 0. In [14], the author and Watanabe studied the Kolmogorov diffusion $(X(t), Y(t))$ on $\mathbb{R}^2$ starting at (x, y) where we define $Y(t) = y + B(t)$ and $X(t) = x + \int_0^t Y(s)\mathrm{d}s$.

The following estimates are obtained concerning the first hitting time of a half-line $\tau = \inf\{t \geq 0 | X(t) = 0, Y(t) \geq 0\}$:

$$1 - E_{(x,y)}\left[\mathrm{e}^{-\mu_1 s^2 Y(\tau) - \mu_2 s^4 \tau}\right] \sim C_4(\mu_1, \mu_2; x, y)s, \quad \text{as } s \to +0,$$

$$P_{(x,y)}[\tau > T] \sim \Gamma(3/4)^{-1} C_4(0, 1; x, y) T^{-1/4}, \quad \text{as } T \to +\infty.$$

See [14] for integral expression of $C_4(\mu_1, \mu_2; x, y)$.

The author and Kotani [13] generalized the result in [14] as follows. Let $a > -1$, $\nu = 1/(a + 2)$, $b > 0$, and define the function $V(x)$ on $\mathbb{R}^2$ by

$$V(x) = x^a \quad \text{for } x > 0; \qquad V(0) = 0; \qquad V(x) = -|x|^a/b \quad \text{for } x < 0.$$

Let $\rho \in (0, 1)$ be the unique solution to $b^\nu \sin(\pi\nu(1 - \rho)) = \sin(\pi\nu\rho)$.

If we set $Y(t) = y + B(t)$ and $X(t) = x + \int_0^t V(Y(s))\mathrm{d}s$, then the process $(X(t), Y(t))$ is a diffusion on $\mathbb{R}^2$ starting at (x, y).

The following estimate is obtained concerning the first hitting time of a half-line $\tau = \inf\{t \geq 0 | X(t) = 0, Y(t) \geq 0\}$:

$$P_{(x,y)}[\tau > T] \sim C(x, y) T^{-\rho/2}, \quad \text{as } T \to \infty,$$

where $C(x, y)$ is given in [13].

The behaviors of the processes studied in [14,13], and Theorem D are different but they admit estimates that bear some resemblance. Particularly, the estimates in [14] and Theorem D both yield $T^{-1/4}$ for the tail of the distribution of the first hitting time.

Let $1 < \alpha \leq 2$. In the present paper, we are mainly concerned with the α-stable Lévy process $(X(t), Y(t))$ with rotational symmetry on $\mathbb{R}^2$ starting from $(x_0, y_0) \in \mathbb{R}^2$. Its probability and expectation are denoted by $P_{(x_0,y_0)}$ and $E_{(x_0,y_0)}$, respectively, and are determined by $E_{(0,0)}[\mathrm{e}^{\mathrm{i}\xi_1 X(t) + \mathrm{i}\xi_2 Y(t)}] = \mathrm{e}^{-t(\xi_1^2+\xi_2^2)^{\alpha/2}}$ for $(\xi_1, \xi_2) \in \mathbb{R}^2$. Let $L_Y(t)$ be the local time at 0 for $Y(\cdot)$: $L_Y(t) = \lim_{\varepsilon \to +0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(Y(s))\mathrm{d}s$.

For $a \in \mathbb{R}$, we set

$$\tau(a) = \inf\{t \geq 0 | Y(t) = 0, X(t) \geq a\}. \tag{1.4}$$

We also set $\Phi_\alpha(\xi_1, \mu_2) = 2\pi / \int_{\mathbb{R}} \frac{\mathrm{d}\xi_2}{\mu_2 + (\xi_1^2+\xi_2^2)^{\alpha/2}}$, $C_1(\alpha) = \Phi_\alpha(1, 0) = 2\pi / B\left(\frac{1}{2}, \frac{\alpha-1}{2}\right)$, $C_2(\alpha) = \Phi_\alpha(0, 1) = \alpha \sin\frac{\pi}{\alpha}$, and

$$I_\alpha(\mu_0, \mu_1, \mu_2) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi(t^2 + 1)} \log(\mu_0 + \Phi_\alpha(\mu_1 t, \mu_2)) \tag{1.5}$$

for $\xi_1 \in \mathbb{R}$ and $\mu_i \geq 0$ $(i = 0, 1, 2)$ such that $\mu_0 + \mu_1 + \mu_2 > 0$.

To state the main theorem, we introduce a family of holomorphic functions. Let $\mathbb{C}_+ = \{z \in \mathbb{C} | \Im z > 0\}$, $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} | \Im z \geq 0\}$ and set

$$\varphi_\alpha(z; \mu_0, \mu_2) = \exp\left(\frac{-1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{z}{t^2 - z^2} \log(\mu_0 + \Phi_\alpha(t, \mu_2)) \mathrm{d}t\right) \tag{1.6}$$

for $z \in \mathbb{C}_+$ and $\mu_i \geq 0$ $(i = 0, 2)$ such that $\mu_0 + \mu_2 > 0$. We can extend $\varphi_\alpha(z; \mu_0, \mu_2)$ to $z \in \mathbb{R}$ by continuity, as is shown in Lemma 3.2. We also set

$$\varphi_\alpha(z; 0, 0) = \frac{1}{\sqrt{C_1(\alpha)}}(-\mathrm{i}z)^{-(\alpha-1)/2} \quad \text{for } z \in \overline{\mathbb{C}_+} \setminus \{0\},$$

where we employ the branch of $z \mapsto z^{-(\alpha-1)/2}$ such that $1^{-(\alpha-1)/2} = 1$.

Now we state the main theorem.

Theorem 1.1. *Let $a > 0$, $\mu_i \geq 0$ $(i = 0, 1, 2)$, and $\mu_0 + \mu_1 + \mu_2 > 0$.*

(i) *It holds*

$$\begin{aligned} &E_{(-a,0)}\left[\mathrm{e}^{-\mu_0 L_Y(\tau(0)) - \mu_1 X(\tau(0)) - \mu_2 \tau(0)}\right] \\ &\quad = \mathrm{e}^{\mu_1 a} - \frac{\mathrm{e}^{\mu_1 a}}{\varphi_\alpha(\mathrm{i}\mu_1; \mu_0, \mu_2)} \int_{-\infty}^{\infty} \mathrm{d}\theta \frac{1 - \mathrm{e}^{-\mathrm{i}a(\theta - \mathrm{i}\mu_1)}}{2\pi \mathrm{i}(\theta - \mathrm{i}\mu_1)} \varphi_\alpha(\theta; \mu_0, \mu_2). \end{aligned}$$

(ii) *As $s \to +0$,*

$$\begin{aligned} &1 - E_{(-a,0)}\left[\mathrm{e}^{-\mu_0 s^{2(\alpha-1)} L_Y(\tau(0)) - \mu_1 s^2 X(\tau(0)) - \mu_2 s^{2\alpha} \tau(0)}\right] \\ &\quad \sim \frac{\exp(I_\alpha(\mu_0, \mu_1, \mu_2)) a^{(\alpha-1)/2}}{\sqrt{C_1(\alpha)}\,\Gamma\left(1 + \frac{\alpha-1}{2}\right)} s^{\alpha-1}, \end{aligned}$$

where $\sim$ means that the ratio between both sides converges to 1.

Since the method of proof applies not only to α-stable Lévy processes with rotational symmetry but to α-stable Lévy processes satisfying the condition that $(X(t) - X(0), Y(t) - Y(0))$ has the same law as $(X(0) - X(t), Y(0) - Y(t))$, we give in Section 3 the proof for such processes.

In Section 4, we state a similar result of Theorem 1.1 for such $(X(t), Y(t))$ that $X(t)$ and $Y(t)$ are independent, $X(t)$ is symmetric β-stable, and $Y(t)$ is symmetric α-stable.

We could not find the definite integral $I_\alpha(\mu_0, \mu_1, \mu_2)$ defined in (1.5) in the huge volume [9] but some marginal values can be evaluated, e.g., $\exp(I_\alpha(0, \mu_1, 0)) = \sqrt{C_1(\alpha)}\mu_1^{(\alpha-1)/2}$ and $\exp(I_\alpha(0, 0, \mu_2)) = \sqrt{C_2(\alpha)}\mu_2^{(\alpha-1)/\alpha}$ using $\Phi_\alpha(\xi_1, 0) = C_1(\alpha)|\xi_1|^{\alpha-1}$ and $\Phi_\alpha(0, \mu_2) = C_2(\alpha)$ $\mu_2^{(\alpha-1)/\alpha}$, respectively.

Corollary 1.2. *Let $y_0 \neq 0$ or $x_0 < 0$.*

(i) *We have, as $s \to +0$,*

$$\begin{aligned} &1 - E_{(x_0,y_0)}\left[\mathrm{e}^{-\mu_0 s^{2(\alpha-1)} L_Y(\tau(0)) - \mu_1 s^2 X(\tau(0)) - \mu_2 s^{2\alpha} \tau(0)}\right] \\ &\quad \sim s^{\alpha-1} \frac{\exp(I_\alpha(\mu_0, \mu_1, \mu_2))}{\sqrt{C_1(\alpha)}\,\Gamma\left(1 + \frac{\alpha-1}{2}\right)} \int_{-\infty}^{-x_0/|y_0|} \frac{(1 + t^2)^{-\alpha/2}}{B\left(\frac{1}{2}, \frac{\alpha-1}{2}\right)} |x_0 + |y_0| t|^{(\alpha-1)/2} \mathrm{d}t. \end{aligned}$$

(ii) *We have, as* $T \to +\infty$,

$$P_{(x_0,y_0)}\left[\tau(0) > T\right] \sim \frac{T^{-(\alpha-1)/(2\alpha)}\sqrt{C_2(\alpha)}}{\sqrt{C_1(\alpha)}\Gamma\left(1+\frac{\alpha-1}{2}\right)\Gamma\left(1-\frac{\alpha-1}{2\alpha}\right)} \times \int_{-\infty}^{-x_0/|y_0|} \frac{(1+t^2)^{-\alpha/2}}{B\left(\frac{1}{2},\frac{\alpha-1}{2}\right)} |x_0+|y_0|t|^{(\alpha-1)/2}\,\mathrm{d}t.$$

Remark 1.2. We can deduce from Bogdan et al. [4] and Bogdan et al. [5] a partial check for Corollary 1.2(ii). If we set $V = \mathbb{R} \setminus \{x \geq 0, y = 0\}$, $\tau(0)$ is the first exit time from V. According to [4], the domain V is $(1/2, r)$-fat for every $r > 0$.

In what follows, the notation $\approx_{C(\alpha)}$ means that the ratio between both sides lies in $(1/C, C)$ with some constant $C > 1$ that depends only on α. Let $\delta_V(\boldsymbol{x})$ be the distance from $\boldsymbol{x} \in V$ to the boundary of V, and let $M_V(\boldsymbol{x})$ be the Martin kernel with the pole at infinity for V; See [5, Lemma 3.3] for the exact form of $M_V(\boldsymbol{x})$. It is elementary to deduce that $M_V(\boldsymbol{x}) \approx_{C(\alpha)} \delta_V(\boldsymbol{x})^{\alpha-1}|\boldsymbol{x}|^{-(\alpha-1)/2}$ for any $\boldsymbol{x} \in V$. It is also elementary to verify

$$\int_{-\infty}^{-x/|y|} (1+t^2)^{-\alpha/2}\, |x+|y|t|^{(\alpha-1)/2}\,\mathrm{d}t \approx_{C(\alpha)} M_V(\boldsymbol{x}) \tag{1.7}$$

for any $\boldsymbol{x} \equiv (x, y) \in V$.

For $\boldsymbol{x} = (x, y)$, we set $A_{t^{1/\alpha}}(\boldsymbol{x}) = \left(x, t^{1/\alpha}/2 + y\right)$ if $y \geq 0$ and $A_{t^{1/\alpha}}(\boldsymbol{x}) = \left(x, -t^{1/\alpha}/2 + y\right)$ if $y < 0$. From theorem 2 in [4], we have

$$P_{(x,y)}\left[\tau(0) > T\right] \approx_{C(\alpha)} \frac{M_V(\boldsymbol{x})}{M_V(A_{T^{1/\alpha}}(\boldsymbol{x}))} \quad \text{for any } \boldsymbol{x} \in V$$

and hence $P_{(x,y)}\left[\tau(0) > T\right] \approx_{C(\alpha)} M_V(\boldsymbol{x})T^{-(\alpha-1)/(2\alpha)}$ as $T \to +\infty$ for any $\boldsymbol{x} = (x, y) \in V$. In view of (1.7), this is in accordance with Corollary 1.2(ii).

In the course of the proof of Theorem 1.1, we obtain explicit formulas in Theorem 1.3 for the first hitting distribution of a line.

The law of a Lévy process $(X(t), Y(t))$ on $\mathbb{R}^2$ is determined by the characteristic exponent $\Psi(\xi_1, \xi_2)$ satisfying $E_{(0,0)}[\mathrm{e}^{\mathrm{i}\xi_1 X(t)+\mathrm{i}\xi_2 Y(t)}] = \mathrm{e}^{-t\Psi(\xi_1,\xi_2)}$ for $(\xi_1, \xi_2) \in \mathbb{R}^2$. If $(X(t), Y(t))$ is a symmetric Markov process (see e.g. [2]) in the sense that $(X(t) - X(0), Y(t) - Y(0))$ has the same law as $(X(0) - X(t), Y(0) - Y(t))$, we have $\Psi(\xi_1, \xi_2) = \Psi(-\xi_1, -\xi_2)$ and hence Ψ is real-valued. We say the process is genuinely two-dimensional if, for any $t > 0$, the distribution of $(X(t), Y(t))$ is not supported on any one-dimensional affine subspace of $\mathbb{R}^2$.

Theorem 1.3. *Set* $T_0^Y := \inf\{t \geq 0 | Y(t) = 0\}$.

(i) *Let* $(X(t), Y(t))$ *be an* α*-stable Lévy process with rotational symmetry on* $\mathbb{R}^2$ *and* $\mathcal{C}$ *be a real random variable such that* $P\left[\mathcal{C} \in \mathrm{d}x\right] = B\left(\frac{1}{2}, \frac{\alpha-1}{2}\right)^{-1}(1+x^2)^{-\alpha/2}\mathrm{d}x$.
Then $P_{(x_0,y_0)}\left[X(T_0^Y) \in \mathrm{d}x\right] = P\left[y_0\mathcal{C} + x_0 \in \mathrm{d}x\right]$.

(ii) *More generally, assume that* $(X(t), Y(t))$ *is a genuinely two-dimensional* α*-stable Lévy process such that* $(X(t) - X(0), Y(t) - Y(0))$ *has the same law as* $(X(0) - X(t), Y(0) - Y(t))$*, and* $E_{(0,0)}[\mathrm{e}^{\mathrm{i}\xi_1 X(t)+\mathrm{i}\xi_2 Y(t)}] = \mathrm{e}^{-t\Psi(\xi_1,\xi_2)}$. *Set* $P[\mathcal{C} \in \mathrm{d}x] = (\int_{\mathbb{R}} \Psi(1, t)^{-1}\mathrm{d}t)^{-1}\, \Psi(1, x)^{-1}\mathrm{d}x$.
Then $P_{(x_0,y_0)}\left[X(T_0^Y) \in \mathrm{d}x\right] = P\left[y_0\mathcal{C} + x_0 \in \mathrm{d}x\right]$.

In the proof of Theorem 1.3 given in Section 2, the assumption "α-stable" is indispensable.

Remark 1.3. (1) There are several papers in the literature concerning explicit hitting distribution of sets by multidimensional stable Lévy processes. Blumenthal–Getoor–Ray [3] obtained the first hitting distribution of $\{x \in \mathbb{R}^d \mid |x| > r\}$ and $\{x \in \mathbb{R}^d \mid |x| < r\}$, and Port [17] obtained that of $\{x \in \mathbb{R}^d \mid |x| = r\}$, by an α-stable Lévy process with rotational symmetry, i.e., $\Psi(\xi) = |\xi|^\alpha$ for $\xi \in \mathbb{R}^d$. This process is a special case of the "relativistic" α-stable processes $X_t^{\alpha,m}$, which are defined by $\Psi^{\alpha,m}(\xi) = (|\xi|^2 + m^2)^{\alpha/2} - m$ with $m > 0$, and Byczkowski et al. [6] gives explicitly the hitting distribution of the half space by $X_t^{\alpha,m}$.

(2) Theorem 1.3(ii) is restricted to the case of dimension 2, but needs not the rotational symmetry. Unfortunately, the author has not succeeded in extending our result to the case of dimension 3 or higher.

(3) Let $(X(t), Y(t))$ be an α-stable Lévy process with rotational symmetry on $\mathbb{R}^2$ and

$$T_R = \inf\{t \geq 0 \mid \|(X(t), Y(t)) - (0, R)\| = R\}.$$

Note that, as $R \to \infty$, the intersection of the circle $\left\{\xi \in \mathbb{R}^2 \mid \|\xi - (0, R)\| = R\right\}$ and any compact set comes close to x-axis. We denote by $\sigma_R(\mathrm{d}\xi)$ the uniform probability measure on this circle.

Adopting a result by Port [17, Theorem 3.1], we obtain the distribution of $(X(T_R), Y(T_R))$ under $P_{(x_0,y_0)}$ as follows:

$$\begin{aligned} &P_{(x_0,y_0)}\left[(X(T_R), Y(T_R)) \in \mathrm{d}\xi\right] \\ &\quad = C_5(\alpha)\left|x_0^2 + (y_0 - r)^2 - r^2\right|^{\alpha-1} \|\xi - (x_0, y_0)\|^{-\alpha}\sigma_R(\mathrm{d}\xi), \end{aligned}$$

where $C_5(\alpha)$ is a positive constant that depends only on $\alpha \in (1, 2)$. If we let $R \to \infty$, this measure converges weakly to

$$B(1/2, (\alpha-1)/2)^{-1}|y_0|^{\alpha-1}(y_0^2 + (x - x_0)^2)^{-\alpha/2} m(d(x, y)),$$

where $m(d(x, y))$ is the one-dimensional Lebesgue measure on the x-axis in the plane $\mathbb{R}^2$. This limit distribution is consistent with Theorem 1.3(i).

(4) It seems interesting to compare Theorem 1.3(i) with the formula (5.12) in [18], which concentrates on the one-dimensional symmetric α-stable Lévy process. Let $X(t)$ and $Y(t)$ are independent symmetric α-stable Lévy processes with $1 < \alpha \leq 2$ and $P[\mathcal{C}_\alpha \in \mathrm{d}x] = \frac{\alpha}{2\pi}\sin\left(\frac{\pi}{\alpha}\right)(1 + |x|^\alpha)^{-1}\mathrm{d}x$. Then it is shown that $P_{(x_0,y_0)}\left[X(T_0^Y) \in \mathrm{d}x\right] = P\left[y_0\mathcal{C}_\alpha + x_0 \in \mathrm{d}x\right]$. Our Theorem 1.3(ii) contains this formula: in this case we have $\Psi(\xi_1, \xi_2) = |\xi_1|^\alpha + |\xi_2|^\alpha$ and $P[\mathcal{C} \in \mathrm{d}x] = P[\mathcal{C}_\alpha \in \mathrm{d}x]$. The variable $\mathcal{C}_\alpha$ is called an α-Cauchy variable in [18] since its law reduces to the Cauchy distribution if $\alpha = 2$. Let us also remark that Theorem 1.3(i) and [18, (5.12)] are different stable analogs of the hitting distribution of a line by a two-dimensional standard Brownian motion, namely the Cauchy distribution. A two-dimensional standard Brownian motion has the independent components and is of rotational symmetry. But a two-dimensional symmetric α-stable Lévy process does not have these two properties at the same time. [18] retains independence of components while Theorem 1.3(i) is based on rotational symmetry. We may consider $\mathcal{C}$ as another α-Cauchy variable.

2. Modified resolvents and hitting of a line in the plane

In this section, we introduce the modified resolvents $U(\mathrm{d}y; \xi_1, \mu)$ and its density $u(y; \xi_1, \mu)$ and apply them to determine the joint law of the first hitting time and place of a line. The resolvents $U(\mathrm{d}y;\xi_1,\mu)$ are modified in the sense that they reduce, if $\xi_1 = 0$, to μ-resolvents for a one-dimensional Lévy process $Y(t)$ as in [1, Section I.2].

Let $(X(t), Y(t))$ be a two-dimensional Lévy process starting from $(x_0, y_0) \in \mathbb{R}^2$. Its probability and expectation are denoted by $P_{(x_0,y_0)}$ and $E_{(x_0,y_0)}$, respectively. Let $\mathcal{F}_t$ be the $P_{(x_0,y_0)}$-completion of $\sigma((X(s), Y(s)); s \in [0,t])$. We denote its characteristic exponent by $\Psi(\xi_1,\xi_2)$, i.e. it holds $E_{(0,0)}[\mathrm{e}^{\mathrm{i}\xi_1 X(t)+\mathrm{i}\xi_2 Y(t)}] = \mathrm{e}^{-t\Psi(\xi_1,\xi_2)}$ for $(\xi_1,\xi_2) \in \mathbb{R}^2$.

Assume $\Psi(\xi_1,\xi_2)$ satisfies

$$\int_{\mathbb{R}} \left| \frac{1}{1+\Psi(0,\xi_2)} \right| \mathrm{d}\xi_2 < \infty. \tag{2.1}$$

Then it is well known (see [1, Corollary II.20, Theorem V.1, and Proposition V.2]) that $Y(t)$ admits a local time process $L_Y(y,t) = \lim_{\varepsilon\to+0} \frac{1}{2\varepsilon}\int_0^t 1_{\{|Y(s)-y|<\varepsilon\}}\mathrm{d}s$ and $t \mapsto L_Y(y,t)$ is a.s. continuous.

Note that (2.1) is a bit stronger than the existence of such $L_Y(y,t)$: (2.1) implies that $\int_{\mathbb{R}} \Re \frac{1}{1+\Psi(0,\xi_2)}\mathrm{d}\xi_2 < \infty$ and a single point is regular for itself for $Y(t)$; these conditions are sufficient for the existence of $L_Y(y,t)$ as above. We assume (2.1) in order for resolvent density to exist. The Lévy processes of our interest, such as symmetric α-stable processes with $1 < \alpha \le 2$, satisfy (2.1).

One can show that, for any bounded Borel function $f(y)$ on $\mathbb{R}$,

$$\int_0^u \mathrm{e}^{\mathrm{i}\xi_1 X(t)} f(Y(t))\mathrm{d}t = \int_{\mathbb{R}} \mathrm{d}y\, f(y) \int_0^u \mathrm{e}^{\mathrm{i}\xi_1 X(t)} d_t L_Y(y,t) \tag{2.2}$$

by standard arguments. Set

$$U(\mathrm{d}y;\xi_1,\mu) := E_{(0,0)}\left[\int_0^\infty \mathrm{e}^{\mathrm{i}\xi_1 X(t)-\mu t} 1_{\{Y(t)\in \mathrm{d}y\}}\mathrm{d}t\right], \tag{2.3}$$

$$u(y;\xi_1,\mu) := E_{(0,0)}\left[\int_0^\infty \mathrm{e}^{\mathrm{i}\xi_1 X(t)-\mu t} d_t L_Y(y,t)\right] \tag{2.4}$$

for $\xi_1, y \in \mathbb{R}$ and $\mu > 0$.

Note that these quantities correspond to the following ones in [1] if $\xi_1 = 0$: (2.2) reduces to $\int_0^u f(Y(t))\mathrm{d}t = \int_{\mathbb{R}} \mathrm{d}y\, f(y) L_Y(y,u)$ in [1, Section V.1]; $U(\mathrm{d}y;0,\mu)$ is the μ-resolvent $U^\mu(0,\mathrm{d}y)$ for $Y(t)$ in [1, Section I.2]; and then

$$u(y;0,\mu) = E_{(0,0)}\left[\int_0^\infty \mathrm{e}^{-\mu t} d_t L_Y(y,t)\right] = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{-\mathrm{i}y\xi_2}}{\mu+\Psi(0,\xi_2)}\mathrm{d}\xi_2$$

is the continuous version of the density for $U^\mu(0,\mathrm{d}y)$ in [1, Section II.5].

Lemma 2.1. *Assume (2.1).*

(i) *The function $y \mapsto u(y;\xi_1,\mu)$ is a version of the density for $U(\mathrm{d}y;\xi_1,\mu)$.*

(ii) *Assume $\xi_2 \mapsto 1/|\mu+\Psi(\xi_1,\xi_2)|$ is integrable for any fixed ξ_1. Then we have*

$$u(y;\xi_1,\mu) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{-\mathrm{i}y\xi_2}}{\mu+\Psi(\xi_1,\xi_2)}\mathrm{d}\xi_2. \tag{2.5}$$

Proof. To prove (i), let f be a bounded Borel function on $\mathbb{R}$. It is elementary to deduce that

$$\int_0^u e^{i\xi_1 X(t)} f(Y(t))dt = \int_{\mathbb{R}} dy\, f(y) \int_0^u e^{i\xi_1 X(t)} d_t L_Y(y,t).$$

We take the Laplace transform in u and the expectation of both sides. Since $\int_{\mathbb{R}} dy\, L_Y(y,u) = u$, we may interchange the integrations to have

$$E_{(0,0)}\left[\int_0^\infty e^{i\xi_1 X(t)-\mu t} f(Y(t))dt\right] = \int_{\mathbb{R}} dy\, f(y) E_{(0,0)}\left[\int_0^\infty e^{-\mu t} e^{i\xi_1 X(t)} d_t L_Y(y,t)\right]$$

and the proof of (i) is complete.

Now the statement of (ii) follows from a standard argument based on the Fourier inversion and $\int_{\mathbb{R}} e^{i\xi_2 y} U(dy;\xi_1,\mu) = \frac{1}{\mu+\Psi(\xi_1,\xi_2)}$. $\square$

Note that if $1/|\mu + \Psi(\xi_1,\xi_2)|$ is integrable with respect to $d\xi_2$ for some $\mu > 0$, then it is integrable for any $\mu > 0$.

Note also that $1/|\mu + \Psi(\xi_1,\xi_2)|$ is integrable with respect to $d\xi_2$ if the process is genuinely two-dimensional and $\forall c > 0, \forall(\xi_1,\xi_2)$, $\Psi(c\xi_1, c\xi_2) = c^\alpha \Psi(\xi_1,\xi_2)$. Indeed, $\Re\Psi(\xi_1,\xi_2) \geq 0$, Ψ vanishes only at $(0,0)$, and we have $1/|\mu + \Psi(\xi_1,\xi_2)| \sim |\xi_2|^{-\alpha}/|\Psi(\xi_1/\xi_2, 1)| \sim |\xi_2|^{-\alpha}/|\Psi(0,1)|$ as $\xi_2 \to \infty$. A similar bound holds when $\xi_2 \to -\infty$.

We next set, for any fixed $\xi_1 \in \mathbb{R}$ and $\mu > 0$,

$$N(t) = e^{i\xi_1 X(t)-\mu t} u(-Y(t);\xi_1,\mu). \tag{2.6}$$

This process is bounded since $|u(y;\xi_1,\mu)| \leq u(0;0,\mu)$ by (2.4).

Lemma 2.2. *Assume (2.1). Then for any starting point $(x_0,y_0) \in \mathbb{R}^2$, under $P_{(x_0,y_0)}$,*

(i) *$N(t) + \int_0^t e^{i\xi_1 X(s)-\mu s} d_s L_Y(0,s)$ is a uniformly integrable martingale;*
(ii) *$M(t) = e^{L_Y(0,t)/u(0;\xi_1,\mu)} N(t)$ is a local martingale.*

Proof. (i) By the translation invariance, we have from (2.4)

$$u(-y;\xi_1,\mu) = E_{(0,y)}\left[\int_0^\infty e^{i\xi_1 X(s)-\mu s} d_s L_Y(0,s)\right]$$

and then we may obtain $N(t) = E_{(x_0,y_0)}\left[\int_t^\infty e^{i\xi_1 X(s)-\mu s} d_s L_Y(0,s) | \mathcal{F}_t\right]$. Indeed, conditionally on $(X(t),Y(t)) = (x,y)$, $((X(t+u)-x, Y(t+u))\,;\, u \geq 0)$ is independent of $\mathcal{F}_t$ and has the law $P_{(0,y)}$. Hence we obtain

$$\begin{aligned}
&E_{(x_0,y_0)}\left[\int_t^\infty e^{i\xi_1 X(s)-\mu s} d_s L_Y(0,s) | \mathcal{F}_t\right]\\
&= e^{i\xi_1 X(t)-\mu t} E_{(x_0,y_0)}\left[\int_t^\infty e^{i\xi_1 (X(s)-X(t))-\mu(s-t)} d_s L_Y(0,s) | \mathcal{F}_t\right]\\
&= e^{i\xi_1 X(t)-\mu t} E_{(0,Y(t))}\left[\int_0^\infty e^{i\xi_1 X(u)-\mu u} d_u L_Y(0,u)\right]\\
&= e^{i\xi_1 X(t)-\mu t} u(-Y(t);\xi_1,\mu) = N(t),
\end{aligned}$$

where in the second equality we have changed variables to $u = s - t$.

We define

$$\begin{aligned}\tilde{M}(t) &:= N(t) + \int_0^t \mathrm{e}^{\mathrm{i}\xi_1 X(s) - \mu s} d_s L_Y(0, s) \\ &= E_{(x_0, y_0)} \left[\int_0^\infty \mathrm{e}^{\mathrm{i}\xi_1 X(s) - \mu s} d_s L_Y(0, s) | \mathcal{F}_t \right],\end{aligned}$$

which is clearly a uniformly integrable martingale.

(ii) The stochastic differential of $N(t)$ is $\mathrm{d}N(t) = \mathrm{d}\tilde{M}(t) - \mathrm{e}^{\mathrm{i}\xi_1 X(t) - \mu t} d_t L_Y(0, t)$. Since $t \mapsto L_Y(0, t)$ is a continuous increasing process, $\mathrm{d}M(t)$ is expressed as follows.

$$\begin{aligned}\mathrm{d}M(t) &= \mathrm{e}^{L_Y(0,t)/u(0;\xi_1,\mu)} \mathrm{d}N(t) + N(t) d \left(\mathrm{e}^{L_Y(0,t)/u(0;\xi_1,\mu)} \right) \\ &= \mathrm{e}^{L_Y(0,t)/u(0;\xi_1,\mu)} \left(\mathrm{d}N(t) + \frac{N(t)}{u(0; \xi_1, \mu)} d_t L_Y(0, t) \right) \\ &= \mathrm{e}^{L_Y(0,t)/u(0;\xi_1,\mu)} \Bigg(\mathrm{d}\tilde{M}(t) - \mathrm{e}^{\mathrm{i}\xi_1 X(t) - \mu t} d_t L_Y(0, t) \\ &\quad + \frac{\mathrm{e}^{\mathrm{i}\xi_1 X(t) - \mu t} u(-Y(t); \xi_1, \mu)}{u(0; \xi_1, \mu)} d_t L_Y(0, t) \Bigg) \\ &= \mathrm{e}^{L_Y(0,t)/u(0;\xi_1,\mu)} \mathrm{d}\tilde{M}(t).\end{aligned}$$

In the last line, we used the fact that $Y(t) = 0$ on the support of $d_t L_Y(0, t)$. For the differential calculus for semimartingales, we refer the reader to [7], in particular Theorem 12.21 therein. Since $\tilde{M}(t)$ is a martingale and $\mathrm{e}^{L_Y(0,t)/u(0;\xi_1,\mu)}$ is locally bounded, $M(t)$ is a local martingale. □

Let

$$L_Y^{-1}(t) := \inf\{s \geq 0 | L_Y(0, s) > t\} \quad \text{and} \quad \Xi(t) = X(L_Y^{-1}(t)). \tag{2.7}$$

Then, under $P_{(x_0,0)}$, $(\Xi(t), L_Y^{-1}(t))$ is a two-dimensional Lévy process starting from $(x_0, 0)$.

Lemma 2.3. *Assume (2.1) and the condition in Lemma 2.1(ii). Then the Lévy process* $(\Xi(t), L_Y^{-1}(t))$ *has the following Fourier–Laplace transform:*

$$E_{(0,0)}[\mathrm{e}^{\mathrm{i}\xi_1 \Xi(t) - \mu L_Y^{-1}(t)}] = \mathrm{e}^{-t\Phi(\xi_1,\mu)} \tag{2.8}$$

with the Fourier–Laplace characteristic exponent

$$\Phi(\xi_1, \mu) = 2\pi \Big/ \int_{\mathbb{R}} \frac{1}{\mu + \Psi(\xi_1, \xi_2)} \mathrm{d}\xi_2 \quad \text{for } \xi_1 \in \mathbb{R} \text{ and } \mu > 0. \tag{2.9}$$

If $(X(t), Y(t))$ *is a genuinely two-dimensional symmetric* α*-stable Lévy process, (2.9) is also valid for* $\xi_1 \neq 0$ *and* $\mu = 0$.

Proof. If $\mu > 0$, we stop $M(t)$ at $L_Y^{-1}(t)$ to obtain a bounded martingale. Then we have

$$E_{(0,0)} \left[\mathrm{e}^{(1/u(0;\xi_1,\mu))t} \mathrm{e}^{-\mu L_Y^{-1}(t) + \mathrm{i}\xi_1 X(L_Y^{-1}(t))} u(0; \xi_1, \mu) \right] = M(0) = u(0; \xi_1, \mu),$$

which implies (2.9) by (2.5).

Fix $\xi_1 \neq 0$. If $(X(t), Y(t))$ is a genuinely two-dimensional symmetric α-stable Lévy process, we have $\inf_{\xi_2 \in \mathbb{R}} \Psi(\xi_1, \xi_2) > 0$ and $\Psi(\xi_1, \xi_2) \sim |\xi_2|^\alpha \Psi(0, 1)$ as $|\xi_2| \to \infty$. The condition in Lemma 2.1(ii) is satisfied, as is seen in the arguments following the proof of Lemma 2.1. On one hand, by the dominated convergence, $\lim_{\mu \to +0} \Phi(\xi_1, \mu) = 2\pi / \int_{\mathbb{R}} \frac{1}{\Psi(\xi_1, \xi_2)} d\xi_2$. On the other hand, $E_{(0,0)}[e^{i\xi_1 \Xi(t)}] = \lim_{\mu \to +0} E_{(0,0)}[e^{i\xi_1 \Xi(t) - \mu L_Y^{-1}(t)}] = \exp\left(-t \lim_{\mu \to +0} \Phi(\xi_1, \mu)\right)$. □

We will determine the joint law of the first hitting time/place of a line.

For any $a \in \mathbb{R}$, set

$$T_a^Y = \inf\{t \geq 0 | Y(t) = a\},$$

which is the first hitting time of a line $\{(x, a) | x \in \mathbb{R}\}$ by (X, Y). The following lemma is an extension of a well-known fact, see e.g. Corollary II.18 in [1].

Lemma 2.4. *Assume (2.1) and let $\xi_1 \in \mathbb{R}$, $\mu > 0$, and $a \neq 0$. Then* $E_{(0,0)}\left[e^{i\xi_1 X(T_a^Y) - \mu T_a^Y}\right] = \frac{u(a; \xi_1, \mu)}{u(0; \xi_1, \mu)}$.

Proof. Let our process start from $(0, -a)$ and we stop $M(t)$ at T_0^Y. Since $L_Y(0, T_0^Y) = 0$, we have by (2.6) $M(T_0^Y) = N(T_0^Y) = e^{i\xi_1 X(T_0^Y) - \mu T_0^Y} u(0; \xi_1, \mu)$ and

$$E_{(0,-a)}\left[e^{i\xi_1 X(T_0^Y) - \mu T_0^Y}\right] = \frac{u(a; \xi_1, \mu)}{u(0; \xi_1, \mu)}.$$

By the translation invariance, we have the statement. □

Proof (Theorem 1.3)**.** Fix $\xi_1 > 0$. By the same argument as in the proof of Lemma 2.3, we have

$$u(y; \xi_1, 0) := \lim_{\mu \to +0} u(y; \xi_1, \mu) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-iy\xi_2}}{\Psi(\xi_1, \xi_2)} d\xi_2.$$

Since $\xi_1 \neq 0$, we have $\Psi(\xi_1, \xi_2) > 0$ for any $\xi_2 \in \mathbb{R}$ and hence $u(0; \xi_1, 0) \in (0, \infty)$. Stopping $M(t)$ at T_0^Y, we have

$$E_{(x_0,y_0)}\left[e^{i\xi_1 X(T_0^Y) - \mu T_0^Y}\right] = \frac{e^{i\xi_1 x_0} u(-y_0; \xi_1, \mu)}{u(0; \xi_1, \mu)}$$

by (2.6). We let $\mu \to +0$ to obtain

$$E_{(x_0,y_0)}\left[e^{i\xi_1 X(T_0^Y)}\right] = \frac{e^{i\xi_1 x_0} u(-y_0; \xi_1, 0)}{u(0; \xi_1, 0)}. \tag{2.10}$$

By substituting $\xi_2 = \xi_1 x$, we have

$$u(y; \xi_1, 0) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-iy\xi_1 x}}{\Psi(\xi_1, \xi_1 x)} \xi_1 dx = \frac{\xi_1^{1-\alpha}}{2\pi} \int_{\mathbb{R}} \frac{e^{-iy\xi_1 x}}{\Psi(1, x)} dx$$

since $\Psi(c\xi_1, c\xi_2) = c^\alpha \Psi(\xi_1, \xi_2)$. Putting this into (2.10), we have

$$\begin{aligned} E_{(x_0,y_0)}\left[e^{i\xi_1 X(T_0^Y)}\right] &= \left(\int_{\mathbb{R}} \frac{1}{\Psi(1, t)} dt\right)^{-1} \int_{\mathbb{R}} \frac{e^{i\xi_1 y_0 x + i\xi_1 x_0}}{\Psi(1, x)} dx \\ &= \int_{\mathbb{R}} e^{i\xi_1 (y_0 x + x_0)} \left(\int_{\mathbb{R}} \frac{1}{\Psi(1, t)} dt\right)^{-1} \Psi(1, x)^{-1} dx. \end{aligned}$$

The complex conjugation of both sides yields the same formula for $\xi_1 < 0$. This determines the distribution of $X(T_0^Y)$ since the right hand side is equal to $E[\exp(\mathrm{i}\xi_1(y_0\mathcal{C} + x_0))]$, where $P[\mathcal{C} \in \mathrm{d}x] = \left(\int_{\mathbb{R}} \Psi(1,t)^{-1}\mathrm{d}t\right)^{-1} \Psi(1,x)^{-1}\mathrm{d}x$. $\square$

3. Proof of Theorem 1.1

In Section 1 we assumed that $(X(t), Y(t))$ is of rotational symmetry but the proof of Theorem 1.1 carries over to other stable Lévy processes. So we assume, in this section, that $(X(t), Y(t))$ is a genuinely two-dimensional symmetric α-stable Lévy process. In other words, $\Psi(\xi_1, \xi_2)$ vanishes only at $(0, 0)$, $\Psi(\xi_1, \xi_2) = \Psi(-\xi_1, -\xi_2) \in \mathbb{R}$, and

$$\forall c > 0, \quad \Psi(c\xi_1, c\xi_2) = c^\alpha \Psi(\xi_1, \xi_2) \in \mathbb{R}.$$

Hence $(X(t), Y(t))$ satisfies the following:

$$\left((-X(t), -Y(t))_{t\ge 0};\, P_{(x,y)}\right) \overset{\text{law}}{=} \left((X(t), Y(t))_{t\ge 0};\, P_{(-x,-y)}\right),$$

$$\left((X(ct), Y(ct))_{t\ge 0};\, P_{(c^{1/\alpha}x, c^{1/\alpha}y)}\right) \overset{\text{law}}{=} \left((c^{1/\alpha}X(t), c^{1/\alpha}Y(t))_{t\ge 0};\, P_{(x,y)}\right), \tag{3.1}$$

for any $c > 0$ and $(x, y) \in \mathbb{R}^2$.

Recall that $\Xi(t) = X(L_Y^{-1}(t))$. The process $(\Xi(t), L_Y^{-1}(t))$ inherits the symmetry and the scaling property from $(X(t), Y(t))$: it holds, for any $c > 0$ and $x \in \mathbb{R}$,

$$\left((-\Xi(t), L_Y^{-1}(t))_{t\ge 0};\, P_{(x,0)}\right) \overset{\text{law}}{=} \left((\Xi(t), L_Y^{-1}(t))_{t\ge 0};\, P_{(-x,0)}\right),$$

$$\left((\Xi(c^{\alpha-1}t), L_Y^{-1}(c^{\alpha-1}t))_{t\ge 0};\, P_{(cx,0)}\right) \overset{\text{law}}{=} \left((c\Xi(t), c^\alpha L_Y^{-1}(t))_{t\ge 0};\, P_{(x,0)}\right). \tag{3.2}$$

This process also obeys the following translation invariance:

$$\left((\Xi(t), L_Y^{-1}(t))_{t\ge 0};\, P_{(x,0)}\right) \overset{\text{law}}{=} \left((\Xi(t) + x, L_Y^{-1}(t))_{t\ge 0};\, P_{(0,0)}\right). \tag{3.3}$$

It is possible to state the symmetry and the scaling property in terms of the Fourier–Laplace characteristic exponent Φ in (2.8): we have $\Phi(\xi_1, \mu_2) = \Phi(-\xi_1, \mu_2) \in \mathbb{R}$ and $\Phi(c\xi_1, c^\alpha\mu_2) = c^{\alpha-1}\Phi(\xi_1, \mu_2)$.

The marginal values of Φ are the following: $\Phi(0, \mu_2) = C_2(\alpha)\Psi(0, 1)^{1/\alpha}\mu_2^{(\alpha-1)/\alpha}$ with $C_2(\alpha) = 2\pi / \int_{\mathbb{R}} \frac{\mathrm{d}x}{1+|x|^\alpha} = \alpha \sin\left(\frac{\pi}{\alpha}\right)$; $\Phi(\xi_1, 0) = \Phi(1, 0)|\xi_1|^{\alpha-1}$ with $\Phi(1, 0) = 2\pi / \int_{\mathbb{R}} \frac{\mathrm{d}x}{\Psi(1,x)}$. If $\Psi(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2)^{\alpha/2}$, we have $\Phi(1, 0) = C_1(\alpha) = 2\pi / B\left(\frac{1}{2}, \frac{\alpha-1}{2}\right)$ by Gradshteyn–Ryzhik [9], formula 3-251-2 in p. 297.

Let $\tau(a) = \inf\{t \ge 0 | Y(t) = 0, X(t) \ge a\}$ and $\sigma(a) = \inf\{t \ge 0 | \Xi(t) \ge a\}$ for $a \in \mathbb{R}$. By (3.1) and (3.2), we have

$$(\tau(ca), X(\tau(ca))) \overset{\text{law}}{=} \left(c^\alpha \tau(ca), cX(\tau(ca))\right)$$

$$\left(\sigma(ca), \Xi(\sigma(ca)), L_Y^{-1}(\sigma(ca))\right) \overset{\text{law}}{=} \left(c^{\alpha-1}\sigma(a), c\Xi(\sigma(a)), c^\alpha L_Y^{-1}(\sigma(a))\right) \tag{3.4}$$

under $P_{(0,0)}$ for any $c > 0$.

Note that $\sigma(a) = L_Y(\tau(a))$, $\Xi(\sigma(a)) = X(\tau(a))$, and $L_Y^{-1}(\sigma(a)) = \tau(a)$. Hence the hitting time of interest, $\tau(a)$, can be studied via $\sigma(a)$ and its companions. In particular, we resort to the fluctuation theory in [10] for the Lévy process $(\Xi(t), L_Y^{-1}(t))$ in Lemma 3.2.

We also set $\overline{\Xi}(t) = \sup_{0\le u\le t} \Xi(u)$.

Lemma 3.1. *Let $a > 0$, $\mu_i \ge 0$ $(i = 0, 1, 2)$, and $\mu_0 + \mu_1 + \mu_2 > 0$. Then*

$$\left(1 - E_{(0,0)}\left[\mathrm{e}^{-\mu_0\sigma(a)-\mu_1\Xi(\sigma(a))-\mu_2L_Y^{-1}(\sigma(a))}\right]\right)\left(\int_0^\infty \mathrm{d}t\ E_{(0,0)}\left[\mathrm{e}^{-\mu_0 t-\mu_1\overline{\Xi}(t)-\mu_2L_Y^{-1}(t)}\right]\right)$$
$$= \int_0^\infty \mathrm{d}t\ E_{(0,0)}\left[\mathrm{e}^{-\mu_0 t-\mu_1\overline{\Xi}(t)-\mu_2L_Y^{-1}(t)}; \overline{\Xi}(t) < a\right].$$

Proof. We have only to use the strong Markov property of $(\Xi(t), L_Y^{-1}(t))$ at $\sigma(a)$. Note that $\overline{\Xi}(\sigma(a)) = \Xi(\sigma(a))$ by the right-continuity of the path. $\square$

We now redefine the function $\varphi(z; \mu_0, \mu_2)$. If $\Psi(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2)^{\alpha/2}$, the coincidence of two definitions will be shown in Lemma 3.2(iii). Let $\mathbb{C}_+ = \{z \in \mathbb{C} | \Im z > 0\}$, $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} | \Im z \ge 0\}$ and set

$$\varphi(z; \mu_0, \mu_2) = \sqrt{\mu_0 + \Phi(0, \mu_2)} \int_0^\infty \mathrm{d}t\ E_{(0,0)}\left[\mathrm{e}^{-\mu_0 t+\mathrm{i}z\overline{\Xi}(t)-\mu_2L_Y^{-1}(t)}\right] \tag{3.5}$$

for $z \in \overline{\mathbb{C}_+}$ and $\mu_i \ge 0$ $(i = 0, 2)$ such that $\mu_0 + \mu_2 > 0$. For $\mu_0 = \mu_2 = 0$, we set

$$\varphi(z; 0, 0) = \frac{1}{\sqrt{\Phi(1, 0)}}(-\mathrm{i}z)^{-(\alpha-1)/2} \quad \text{for } z \in \overline{\mathbb{C}_+} \setminus \{0\}, \tag{3.6}$$

where we employ the branch of $z \mapsto z^{-(\alpha-1)/2}$ such that $1^{-(\alpha-1)/2} = 1$, or equivalently, $\varphi(i; 0, 0) = \frac{1}{\sqrt{\Phi(1,0)}}$. For $\mu_i \ge 0$ $(i = 0, 1, 2)$ such that $\mu_0 + \mu_1 + \mu_2 > 0$, we define

$$I_\Phi(\mu_0, \mu_1, \mu_2) = \int_{-\infty}^\infty \frac{1}{2\pi(1 + t^2)} \log(\mu_0 + \Phi(\mu_1 t, \mu_2))\mathrm{d}t, \tag{3.7}$$

convergence of which is verified using

$$0 \le \Phi(\xi_1, \mu_2) = |\xi_1|^{\alpha-1}\Phi(1, |\xi_1|^{-\alpha}\mu_2) \sim \Phi(1, 0)|\xi_1|^{\alpha-1}, \tag{3.8}$$

as $|\xi_1|^\alpha/\mu_2 \to +\infty$. If $\mu_0 = \mu_2 = 0$, it is elementary to verify $I_\Phi(0, \mu_1, 0) = \log\left(\sqrt{\Phi(1, 0)}\mu_1^{(\alpha-1)/2}\right)$.

Lemma 3.2. *Let $\mu_i \ge 0$ $(i = 0, 1, 2)$ and $\mu_0 + \mu_2 > 0$. Then the following* (i)–(iv) *hold.*

(i) *The function $z \mapsto \varphi(z; \mu_0, \mu_2)$ is holomorphic on $\mathbb{C}_+$, continuous on $\overline{\mathbb{C}_+}$ and its absolute value is bounded by $\varphi(0; \mu_0, \mu_2) = \frac{1}{\sqrt{\mu_0+\Phi(0,\mu_2)}}$ on $\overline{\mathbb{C}_+}$.*

(ii) *On the positive imaginary axis, we have*

$$\varphi(\mathrm{i}y; \mu_0, \mu_2) = \exp\left(\frac{-1}{2\pi\mathrm{i}}\int_{-\infty}^\infty \frac{\mathrm{i}}{t^2 + 1} \log(\mu_0 + \Phi(yt, \mu_2))\mathrm{d}t\right) \tag{3.9}$$

for $y \ge 0$ and, in an equivalent form,

$$\varphi(\mathrm{i}\mu_1; \mu_0, \mu_2) = \exp(-I_\Phi(\mu_0, \mu_1, \mu_2)).$$

(iii) *For any* $z \in \mathbb{C}_+$,

$$\varphi(z;\mu_0,\mu_2) = \exp\left(\frac{-1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{z}{t^2-z^2}\log(\mu_0+\Phi(t,\mu_2))\mathrm{d}t\right).$$

(iv) *On the real line, we have*

$$|\varphi(\theta;\mu_0,\mu_2)| = \frac{1}{\sqrt{\mu_0+\Phi(\theta,\mu_2)}} \quad \text{for } \theta \in \mathbb{R},$$

and

$$\varphi(\theta;\mu_0,\mu_2) \sim \frac{\exp\left((\operatorname{sgn}\theta)\frac{\pi}{4}(\alpha-1)\mathrm{i}\right)}{\sqrt{\Phi(1,0)}|\theta|^{(\alpha-1)/2}} \quad \text{as } |\theta| \to \infty.$$

Proof. Lemma 2.3 implies $\varphi(0;\mu_0,\mu_2) = \frac{1}{\sqrt{\mu_0+\Phi(0,\mu_2)}}$. The other statements in (i) can be verified by standard arguments.

To prove (ii), we first quote Theorem 1 in [10]: for any $z \in \overline{\mathbb{C}_+}$ and any $\theta \in \mathbb{R}$, it holds

$$\sqrt{\mu_0+\Phi(0,\mu_2)}\varphi(z;\mu_0,\mu_2) = \exp\left(\int_0^\infty \frac{\mathrm{e}^{-\mu_0 t}\mathrm{d}t}{t} E\left[\left(\mathrm{e}^{\mathrm{i}z\Xi(t)}-1\right)\mathrm{e}^{-\mu_2 L_Y^{-1}(t)}; \Xi(t)>0\right]\right), \tag{3.10}$$

$$|\varphi(\theta;\mu_0,\mu_2)|^2 = \varphi(\theta;\mu_0,\mu_2)\varphi(-\theta;\mu_0,\mu_2) = \frac{1}{\mu_0+\Phi(\theta,\mu_2)}. \tag{3.11}$$

If $y = 0$, the formula (3.9) is reduced to the following. On one hand, $\varphi(0;\mu_0,\mu_2) = \frac{1}{\sqrt{\mu_0+\Phi(0,\mu_2)}}$. On the other hand,

$$\frac{-1}{2\pi}\int_{-\infty}^{\infty}\frac{\log(\mu_0+\Phi(0,\mu_2))}{t^2+1}\mathrm{d}t = \frac{-1}{2}\log(\mu_0+\Phi(0,\mu_2))$$

and the equality in (3.9) holds.

To prove (3.9) when $y > 0$, we need

$$\mathrm{e}^{-yx}1_{\{x>0\}} = \lim_{A\to+\infty}\int_{-A}^{A}\mathrm{d}\theta\frac{1}{2\pi\mathrm{i}(\theta-\mathrm{i}y)}\mathrm{e}^{\mathrm{i}\theta x} \quad \text{for } x \neq 0, \tag{3.12}$$

along with

$$\sup_{x\in\mathbb{R},A\geq 2y}\left|\int_{-A}^{A}\mathrm{d}\theta\frac{1}{2\pi\mathrm{i}(\theta-\mathrm{i}y)}\mathrm{e}^{\mathrm{i}\theta x}\right| \leq 2. \tag{3.13}$$

These can be proven easily by, e.g., arguments using the residue theorem. We denote $\lim_{A\to+\infty}\int_{-A}^{A}\mathrm{d}\theta$ by $\int_{-\infty}^{\infty}\mathrm{d}\theta$ in what follows. We also need the Frullani integral:

$$\int_0^\infty \frac{1}{t}\left(\mathrm{e}^{-\alpha t}-\mathrm{e}^{-\beta t}\right)\mathrm{d}t = \log\beta - \log\alpha \quad \text{for } \alpha,\beta > 0. \tag{3.14}$$

By (3.10),

$$\log\left(\sqrt{\mu_0+\Phi(0,\mu_2)}\varphi(\mathrm{i}y;\mu_0,\mu_2)\right) = \int_0^\infty \frac{\mathrm{e}^{-\mu_0 t}\mathrm{d}t}{t}E_{(0,0)}\left[\left(\mathrm{e}^{-y\Xi(t)}-1\right)\mathrm{e}^{-\mu_2 L_Y^{-1}(t)}; \Xi(t)>0\right]$$

$$= \int_0^\infty \frac{e^{-\mu_0 t}dt}{t}\Bigg(E_{(0,0)}\left[e^{-y\Xi(t)-\mu_2 L_Y^{-1}(t)};\ \Xi(t)>0\right]$$
$$- E_{(0,0)}\left[e^{-\mu_2 L_Y^{-1}(t)};\ \Xi(t)>0\right]\Bigg).$$

Then by virtue of (3.12), (3.13) and symmetry of $\Xi(t)$, the above is equal to

$$\int_0^\infty \frac{e^{-\mu_0 t}dt}{t}\left(E_{(0,0)}\left[\int_{-\infty}^\infty \frac{e^{i\theta\Xi(t)-\mu_2 L_Y^{-1}(t)}}{2\pi i(\theta - iy)}d\theta\right] - \frac{1}{2}E_{(0,0)}\left[e^{-\mu_2 L_Y^{-1}(t)}\right]\right)$$
$$= \int_0^\infty \frac{e^{-\mu_0 t}dt}{t}\left(\int_{-\infty}^\infty \frac{e^{-t\Phi(\theta,\mu_2)}}{2\pi i(\theta - iy)}d\theta - \frac{1}{2}e^{-t\Phi(0,\mu_2)}\right).$$

Since it is real, the above integral is equal to

$$\int_0^\infty \frac{e^{-\mu_0 t}dt}{t}\int_{-\infty}^\infty \frac{y}{2\pi(y^2+\theta^2)}\left(e^{-t\Phi(\theta,\mu_2)} - e^{-t\Phi(0,\mu_2)}\right)d\theta. \tag{3.15}$$

Here we have used $\int_{-\infty}^\infty \frac{y}{2\pi(y^2+\theta^2)}d\theta = 1/2$ for any $y > 0$. Since $\mu_0 + \Phi(\theta, \mu_2) \ge \mu_0 + \Phi(0, \mu_2) > 0$, the integrand of (3.15) is negative everywhere and the interchange of integrations $\int dt \int d\theta = \int d\theta \int dt$ is justified. In view of (3.14), we have (3.15) is equal to

$$\int_{-\infty}^\infty \frac{y\,d\theta}{2\pi(y^2+\theta^2)}\left(\log(\mu_0+\Phi(0,\mu_2)) - \log(\mu_0+\Phi(\theta,\mu_2))\right)$$
$$= \frac{1}{2}\log(\mu_0+\Phi(0,\mu_2)) - \int_{-\infty}^\infty \frac{y\,d\theta}{2\pi(y^2+\theta^2)}\log(\mu_0+\Phi(\theta,\mu_2)).$$

Rearrangement of terms yields (3.9).

Assertion (iii) follows from (i) and (ii) by the uniqueness theorem for holomorphic functions.

The first statement in (iv) follows from (3.11); hence we have $|\varphi(\theta;\mu_0,\mu_2)| \sim \frac{1}{\sqrt{\Phi(\theta,\mu_2)}} \sim \frac{1}{\sqrt{\Phi(1,0)}|\theta|^{(\alpha-1)/2}}$ as $|\theta| \to \infty$ by (3.8).

To prove that $\lim_{|\theta|\to\infty} \arg\varphi(\theta;\mu_0,\mu_2) = \frac{\pi}{4}(\alpha-1)(\mathrm{sgn}\theta)$, we will use Lemma 3.3. If we set $f(x) = \int_0^x \frac{\sin s}{s}ds$ for $x > 0$ and $f(0) = 0$, the function $f(x)$ on $[0,\infty)$ is bounded, continuous, and satisfies $\lim_{x\to+\infty} f(x) = \pi/2$.

Lemma 3.3. (i) *Assume a function $g(x)$ is non-negative and continuous on $[0,\infty)$ and is continuously differentiable on $(0,\infty)$ with $g'(x) < 0$ for any $x \in (0,\infty)$. Set $G(x) = \int_0^x \frac{\sin s}{s}g(s)ds$ for $x > 0$ and $G(0) = 0$. Then $G(x)$ is continuous on $[0,\infty)$, the limit $\lim_{x\to+\infty} G(x)$ exists, and it holds $\sup_{x\ge0}|G(x)| \le g(0)\sup_{x\ge0}|f(x)|$.*

(ii) *Let $g(x,\theta)$ be a non-negative function on $[0,\infty)\times(0,\infty)$ such that for any $\theta > 0$ it holds that $g(0,\theta) = 1$, $\lim_{x\to+\infty} g(x,\theta) = 0$, and the function $x \mapsto g(x,\theta)$ satisfies the assumption for $g(x)$ in the statement* (i) *above; for any $x \ge 0$ it holds that the function $\theta \mapsto g(x,\theta)$ is nondecreasing on $(0,\infty)$ and $\lim_{\theta\to+\infty} g(x,\theta) = 1$. Set $I(\theta) = \int_0^\infty \frac{\sin s}{s}g(s,\theta)ds$ for $\theta > 0$. Then $\sup_{\theta>0}|I(\theta)| \le \sup_{x\ge0}|f(x)|$ and $\lim_{\theta\to+\infty} I(\theta) = \pi/2$.*

Proof. (i) Integrating by parts, we have $G(x) = f(x)g(x) - \int_0^x f(s)g'(s)ds$, which yields $|G(x)| \le \left(\sup_{t\ge0}|f(t)|\right)\left(g(x) + \int_0^x |g'(s)|ds\right) = g(0)\sup_{t\ge0}|f(t)|$. The limit $\lim_{x\to+\infty} G(x)$ exists since the limit $\lim_{x\to+\infty} g(x)$ exists, $g'(x)$ is integrable on $(0,\infty)$, and $f(x)$ is bounded.

(ii) Part (i) and the assumption $\lim_{x\to+\infty} g(x,\theta) = 0$ implies that $I(\theta)$ is well defined and $I(\theta) = -\int_0^\infty f(s)\frac{\partial g}{\partial s}(s,\theta)\mathrm{d}s$. For $y \in (0,1]$, we define $h(y,\theta)$ by $g(h(y,\theta),\theta) = y$ so that $I(\theta) = \int_{(0,1]} f(h(y,\theta))\mathrm{d}y$ for any fixed $\theta > 0$.

Since $\lim_{\theta\to+\infty} h(y,\theta) = +\infty$ for any $y \in (0,1)$, we have the assertion of (ii) by the bounded convergence theorem. □

We resume proving Lemma 3.2(iv). By (3.10) and (3.2), we have

$$\begin{aligned}
\arg\varphi(\theta;\mu_0,\mu_2) &= \Im\left(\int_0^\infty \frac{\mathrm{e}^{-\mu_0 t}\mathrm{d}t}{t} E_{(0,0)}\left[\left(\mathrm{e}^{\mathrm{i}\theta\Xi(t)}-1\right)\mathrm{e}^{-\mu_2 L_Y^{-1}(t)};\Xi(t)>0\right]\right)\\
&= \int_0^\infty \frac{\mathrm{e}^{-\mu_0 t}\mathrm{d}t}{t} E_{(0,0)}\left[\sin\left(\theta\Xi(t)\right)\mathrm{e}^{-\mu_2 L_Y^{-1}(t)};\Xi(t)>0\right]\\
&= \int_0^\infty \frac{\mathrm{e}^{-\mu_0 t}\mathrm{d}t}{t} E_{(0,0)}\Big[\sin\left(\theta t^{1/(\alpha-1)}\Xi(1)\right)\\
&\quad\times \exp\left(-\mu_2 t^{\alpha/(\alpha-1)} L_Y^{-1}(1)\right);\Xi(1)>0\Big]\\
&= (\alpha-1)\lim_{\varepsilon\to+0,A\to+\infty}\int_\varepsilon^A du\frac{\mathrm{e}^{-\mu_0(u/|\theta|)^{\alpha-1}}}{u}\\
&\quad\times E_{(0,0)}\left[\sin\left((\mathrm{sgn}\theta)u\Xi(1)\right)\exp\left(-\mu_2(u/|\theta|)^\alpha L_Y^{-1}(1)\right);\Xi(1)>0\right],\\
&= (\alpha-1)(\mathrm{sgn}\,\theta)\lim_{\varepsilon\to+0,A\to+\infty} E_{(0,0)}\Bigg[\int_\varepsilon^A \mathrm{d}u\frac{\mathrm{e}^{-\mu_0(u/|\theta|)^{\alpha-1}}}{u}\\
&\quad\times \sin\left(u\Xi(1)\right)\exp\left(-\mu_2(u/|\theta|)^\alpha L_Y^{-1}(1)\right);\Xi(1)>0\Bigg],
\end{aligned}$$

where we changed variable to $u = |\theta|t^{1/(\alpha-1)}$ in the fourth equality. We next change variable to $s = \Xi(1)u$ for the integral $\int_\varepsilon^A \mathrm{d}u$ in the expectation:

$$\begin{aligned}
&\int_\varepsilon^A \mathrm{d}u\frac{\mathrm{e}^{-\mu_0(u/|\theta|)^{\alpha-1}}}{u}\sin\left(u\Xi(1)\right)\exp\left(-\mu_2(u/|\theta|)^\alpha L_Y^{-1}(1)\right)\\
&\quad= \int_{\varepsilon\Xi(1)}^{A\Xi(1)} \mathrm{d}s\frac{\mathrm{e}^{-\mu_0 s^{\alpha-1}(|\theta|\Xi(1))^{1-\alpha}}}{s}(\sin s)\exp\left(-\mu_2 s^\alpha(|\theta|\Xi(1))^{-\alpha}L_Y^{-1}(1)\right),
\end{aligned}$$

which we denote by $I(\varepsilon,A,|\theta|)$.

We set $g(x,\theta) = \exp\left(-\mu_0 x^{\alpha-1}(\theta\Xi(1))^{1-\alpha} - \mu_2 x^\alpha(\theta\Xi(1))^{-\alpha}L_Y^{-1}(1)\right)$ so that $(x,\theta)\mapsto g(x,\theta)$ satisfies the assumption in Lemma 3.3(ii).

By Lemma 3.3(i), we have $\sup_{0<\varepsilon<A<\infty}|I(\varepsilon,A,|\theta|)| \le 2\sup_{x\ge0}|f(x)|$ and the limit $I(|\theta|) = \lim_{\varepsilon\to+0,A\to+\infty} I(\varepsilon,A,|\theta|)$ exists and hence

$$\arg\varphi(\theta;\mu_0,\mu_2) = (\alpha-1)(\mathrm{sgn}\,\theta)E_{(0,0)}\left[I(|\theta|);\Xi(1)>0\right].$$

From Lemma 3.3(ii), it follows that $I(|\theta|)$ is bounded and converges to $\pi/2$ as $|\theta|\to\infty$, which implies that $\arg\varphi(\theta;\mu_0,\mu_2)$ converges to $(\alpha-1)(\mathrm{sgn}\theta)E_{(0,0)}\left[\pi/2;\Xi(1)>0\right] = \frac{\pi}{4}(\alpha-1)(\mathrm{sgn}\theta)$ by the bounded convergence theorem as $\theta\to+\infty$ or $\theta\to-\infty$. □

Lemma 3.4. *For any $a > 0$ and $\mu_i \geq 0$ ($i = 0, 1, 2$) such that $\mu_0 + \mu_1 + \mu_2 > 0$, it holds*

$$1 - E_{(0,0)}\left[\mathrm{e}^{-\mu_0\sigma(a)-\mu_1\Xi(\sigma(a))-\mu_2 L_Y^{-1}(\sigma(a))}\right]$$
$$= \frac{1}{\varphi(\mathrm{i}\mu_1;\mu_0,\mu_2)}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}a(\theta-\mathrm{i}\mu_1)}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu_1)}\varphi(\theta;\mu_0,\mu_2).$$

Proof. In view of (3.5) and Lemma 3.1, the statement is equivalent to the following if $\mu_0 + \mu_2 > 0$:

$$\int_0^{\infty}\mathrm{d}t\,E_{(0,0)}\left[\mathrm{e}^{-\mu_0 t-\mu_1\overline{\Xi}(t)-\mu_2 L_Y^{-1}(t)};\,\overline{\Xi}(t) < a\right]$$
$$= \int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}a(\theta-\mathrm{i}\mu_1)}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu_1)}\int_0^{\infty}\mathrm{d}t\,E_{(0,0)}\left[\mathrm{e}^{-\mu_0 t+\mathrm{i}\theta\overline{\Xi}(t)-\mu_2 L_Y^{-1}(t)}\right]. \tag{3.16}$$

By (3.12) and (3.13), we have

$$\mathrm{e}^{-\mu_1 x}1_{\{0<x<a\}} = \lim_{A\to+\infty}\int_{-A}^{A}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}a(\theta-\mathrm{i}\mu_1)}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu_1)}\mathrm{e}^{\mathrm{i}\theta x}\quad \text{for } x \notin \{0, a\}, \tag{3.17}$$

$$\sup_{x\in\mathbb{R},A\geq 2\mu_1\geq 0}\left|\int_{-A}^{A}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}a(\theta-\mathrm{i}\mu_1)}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu_1)}\mathrm{e}^{\mathrm{i}\theta x}\right| \leq 4. \tag{3.18}$$

The measure $\int_0^{\infty}\mathrm{d}t\,P_{(0,0)}[\mathrm{e}^{-\mu_0 t-\mu_2 L_Y^{-1}(t)};\,\overline{\Xi}(t)\in \mathrm{d}x]$ on $\mathbb{R}$ is a finite measure with the total mass $1/(\mu_0+\Phi(0,\mu_2))$. Lemma 3.2(iv) implies that this measure has no point mass. Upon integrating (3.17) with this measure, we obtain (3.16) and the proof for the case $\mu_0 + \mu_2 > 0$ is complete since (3.18) allows us to apply the bounded convergence theorem.

For the case $\mu_0 = \mu_2 = 0$, we replace $\mu_0 > 0$ and $\mu_2 > 0$ with $c^{\alpha-1}\mu_0$ and $c^{\alpha}\mu_2$, respectively, and then let $c \to +0$. So we start with $\mu_i > 0$ ($i = 0, 1, 2$) and

$$1 - E_{(0,0)}\left[\mathrm{e}^{-c^{\alpha-1}\mu_0\sigma(a)-\mu_1\Xi(\sigma(a))-c^{\alpha}\mu_2\eta(\sigma(a))}\right]$$
$$= \frac{1}{\varphi(\mathrm{i}\mu_1;c^{\alpha-1}\mu_0,c^{\alpha}\mu_2)}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}a(\theta-\mathrm{i}\mu_1)}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu_1)}\varphi(\theta;c^{\alpha-1}\mu_0,c^{\alpha}\mu_2).$$

By (3.4), we have

$$\varphi(z;c^{\alpha-1}\mu_0,c^{\alpha}\mu_2) = c^{-(\alpha-1)/2}\varphi(c^{-1}z;\mu_0,\mu_2)$$

for any $c > 0$ and $z \in \overline{\mathbb{C}_+}$. The left hand side above satisfies

$$\varphi(\theta;c^{\alpha-1}\mu_0,c^{\alpha}\mu_2) \leq \frac{c^{-(\alpha-1)/2}}{\sqrt{\mu_0+\Phi(0,\mu_2)}},$$

$$\varphi(\theta;c^{\alpha-1}\mu_0,c^{\alpha}\mu_2) \to \frac{\mathrm{e}^{\mathrm{i}(\operatorname{sgn}\theta)\frac{\pi}{4}(\alpha-1)}}{\sqrt{\Phi(1,0)}|\theta|^{(\alpha-1)/2}}\quad \text{as } c \to +0, \tag{3.19}$$

for any $\theta \in \mathbb{R}$ by Lemma 3.2(iv). Note that the ratio between both sides of (3.19) converges uniformly to 1 in $\theta \in (-\infty, -c^{2/3}) \cup (c^{2/3}, \infty)$. It is then elementary to verify

$$\begin{aligned}
&\lim_{c\to+0}\frac{1}{\varphi(\mathrm{i}\mu_1;c^{\alpha-1}\mu_0,c^{\alpha}\mu_2)}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}a(\theta-\mathrm{i}\mu_1)}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu_1)}\varphi(\theta;c^{\alpha-1}\mu_0,c^{\alpha}\mu_2)\\
&\quad=\sqrt{\Phi(1,0)}\mu_1^{(\alpha-1)/2}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}a(\theta-\mathrm{i}\mu_1)}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu_1)}\cdot\frac{\mathrm{e}^{\mathrm{i}(\operatorname{sgn}\theta)\pi(\alpha-1)/4}}{\sqrt{\Phi(1,0)}|\theta|^{(\alpha-1)/2}}\\
&\quad=\frac{1}{\varphi(\mathrm{i}\mu_1;0,0)}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}a(\theta-\mathrm{i}\mu_1)}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu_1)}\varphi(\theta;0,0),
\end{aligned}\tag{3.20}$$

where we have used

$$\begin{aligned}
\lim_{c\to+0}\log\left(1/\varphi(\mathrm{i}\mu_1;c^{\alpha-1}\mu_0,c^{\alpha}\mu_2)\right)&=\lim_{c\to+0}I_{\Phi}(c^{\alpha-1}\mu_0,\mu_1,c^{\alpha}\mu_2)\\
&=\log(\sqrt{\Phi(1,0)}\mu_1^{(\alpha-1)/2})
\end{aligned}$$

and (3.6). On the other hand,

$$\lim_{c\to+0}E_{(0,0)}\left[\mathrm{e}^{-c^{\alpha-1}\mu_0\sigma(a)-\mu_1\Xi(\sigma(a))-c^{\alpha}\mu_2L_Y^{-1}(\sigma(a))}\right]=E_{(0,0)}\left[\mathrm{e}^{-\mu_1\Xi(\sigma(a))}\right],$$

and hence the statement of this lemma is valid for $\mu_0=\mu_2=0$. □

Remark 3.5. In the terminology of Chapter VI in [1], $\Xi(\sigma(a))-a$ is the overshoot for a one-dimensional symmetric $(\alpha-1)$-stable Lévy process $\Xi(t)$. Adopting Exercise VI.1 and Lemma VIII.1 in [1], we have the following double Laplace transform:

$$\int_0^{\infty}\mathrm{d}a\mathrm{e}^{-qa}\left(1-E_{(0,0)}\left[\mathrm{e}^{-\mu\Xi(\sigma(a))}\right]\right)=\frac{\mu^{(\alpha-1)/2}}{q(q+\mu)^{(\alpha-1)/2}}.\tag{3.21}$$

On the other hand, we take the Laplace transform of (3.20) and replace $\mathrm{e}^{\mathrm{i}(\operatorname{sgn}\theta)\pi(\alpha-1)/4}/|\theta|^{(\alpha-1)/2}$ by $1/(-\mathrm{i}\theta)^{(\alpha-1)/2}$ to obtain

$$\begin{aligned}
&\int_0^{\infty}\mathrm{d}a\mathrm{e}^{-qa}\left(1-E_{(0,0)}\left[\mathrm{e}^{-\mu\Xi(\sigma(a))}\right]\right)\\
&\quad=\int_0^{\infty}\mathrm{d}a\mathrm{e}^{-qa}\mu^{(\alpha-1)/2}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}(\theta-\mathrm{i}\mu)a}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu)}\frac{1}{(-\mathrm{i}\theta)^{(\alpha-1)/2}},
\end{aligned}$$

where we take the branch of $z\mapsto z^{(\alpha-1)/2}$ such that $1^{(\alpha-1)/2}=1$. Since $|\theta-\mathrm{i}\mu|^{-1}|\theta|^{-(\alpha-1)/2}$ is integrable with respect to $\mathrm{d}\theta$, we may interchange the integrations and the above is equal to

$$\begin{aligned}
&\mu^{(\alpha-1)/2}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{\frac{1}{q}-\frac{1}{q+\mu+\mathrm{i}\theta}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu)}\frac{1}{(-\mathrm{i}\theta)^{(\alpha-1)/2}}\\
&\quad=\frac{\mu^{(\alpha-1)/2}}{q}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1}{2\pi\mathrm{i}(\theta-\mathrm{i}(q+\mu))}\frac{1}{(-\mathrm{i}\theta)^{(\alpha-1)/2}},
\end{aligned}$$

where we used $\left(\frac{1}{q}-\frac{1}{q+\mu+\mathrm{i}\theta}\right)\frac{1}{(\theta-\mathrm{i}\mu)}=\frac{1}{q(\theta-\mathrm{i}(q+\mu))}$.

The coincidence with (3.21) is verified by a simple application of the residue theorem.

Lemma 3.6. *For any $a>0$ and $\mu_i\geq 0$ $(i=0,1,2)$ such that $\mu_0+\mu_1+\mu_2>0$, it holds*

$$1-E_{(0,0)}\left[\mathrm{e}^{-\mu_0\sigma(a)-\mu_1\Xi(\sigma(a))-\mu_2L_Y^{-1}(\sigma(a))}\right]$$

$$\sim \frac{\exp(I_\Phi(\mu_0,\mu_1,\mu_2))}{\sqrt{\Phi(1,0)}\Gamma\left(1+\frac{\alpha-1}{2}\right)}a^{(\alpha-1)/2}, \quad \textit{as } a \to +0.$$

Proof. We have $\varphi(\mathrm{i}\mu_1;\mu_0,\mu_2) = \exp(-I_\Phi(\mu_0,\mu_1,\mu_2))$ for any $\mu_i \geq 0$ $(i = 0,1,2)$ such that $\mu_0+\mu_1+\mu_2 > 0$. Indeed, it follows from the definition (3.6) and $I_\Phi(0,\mu_1,0) = \log\left(\sqrt{\Phi(1,0)}\mu_1^{(\alpha-1)/2}\right)$ if $\mu_0=\mu_2=0$; it is shown in Lemma 3.2(ii) for $\mu_0+\mu_2>0$.

Plugging either Lemma 3.2(iv) or the definition (3.6) into Lemma 3.4, we have

$$\begin{aligned}
&1 - E_{(0,0)}\left[\mathrm{e}^{-\mu_0\sigma(a)-\mu_1\Xi(\sigma(a))-\mu_2 L_Y^{-1}(\sigma(a))}\right]\\
&= \frac{1}{\exp(-I_\Phi(\mu_0,\mu_1,\mu_2))}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}a(\theta-\mathrm{i}\mu_1)}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu_1)}\varphi(\theta;\mu_0,\mu_2)\\
&\sim \exp(I_\Phi(\mu_0,\mu_1,\mu_2))\frac{a^{(\alpha-1)/2}}{\sqrt{\Phi(1,0)}}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{1-\mathrm{e}^{-\mathrm{i}x}}{2\pi\mathrm{i}x}|x|^{-(\alpha-1)/2}\mathrm{e}^{(\operatorname{sgn} x)\mathrm{i}\frac{\pi}{4}(\alpha-1)}\\
&= \exp(I_\Phi(\mu_0,\mu_1,\mu_2))\frac{a^{(\alpha-1)/2}}{\sqrt{\Phi(1,0)}\Gamma\left(1+\frac{\alpha-1}{2}\right)}
\end{aligned}$$

as $a \to +0$, where we changed variable to $x = a\theta$. □

Proof (Theorem 1.1)**.** (i) By $L_Y(\tau(0)) = \sigma(0)$, $X(\tau(0)) = \Xi(\sigma(0))$, $\tau(0) = L_Y^{-1}(\sigma(0))$, and by the translation (3.3) of the whole process as well as the half-line to be hit, we have

$$\begin{aligned}
E_{(-a,0)}\left[\mathrm{e}^{-\mu_0 L_Y(\tau(0))-\mu_1 X(\tau(0))-\mu_2\tau(0)}\right] &= E_{(-a,0)}\left[\mathrm{e}^{-\mu_0\sigma(0)-\mu_1\Xi(\sigma(0))-\mu_2 L_Y^{-1}(\sigma(0))}\right]\\
&= E_{(0,0)}\left[\mathrm{e}^{-\mu_0\sigma(a)-\mu_1(\Xi(\sigma(a))-a)-\mu_2 L_Y^{-1}(\sigma(a))}\right]\\
&= \mathrm{e}^{\mu_1 a}E_{(0,0)}\left[\mathrm{e}^{-\mu_0\sigma(a)-\mu_1\Xi(\sigma(a))-\mu_2 L_Y^{-1}(\sigma(a))}\right].
\end{aligned}$$

Then Lemma 3.4 yields the statement.

(ii) Let s be a positive (and small) parameter. Using the scaling property (3.4), we have

$$\begin{aligned}
&1 - E_{(-a,0)}\left[\mathrm{e}^{-\mu_0 s^{2(\alpha-1)}L_Y(\tau(0))-\mu_1 s^2 X(\tau(0))-\mu_2 s^{2\alpha}\tau(0)}\right]\\
&= 1-\mathrm{e}^{\mu_1 s^2 a}E_{(0,0)}\left[\mathrm{e}^{-\mu_0 s^{2(\alpha-1)}\sigma(a)-\mu_1 s^2\Xi(\sigma(a))-\mu_2 s^{2\alpha}L_Y^{-1}(\sigma(a))}\right]\\
&= 1-\mathrm{e}^{\mu_1 s^2 a}E_{(0,0)}\left[\mathrm{e}^{-\mu_0\sigma(s^2a)-\mu_1\Xi(\sigma(s^2a))-\mu_2 L_Y^{-1}(\sigma(s^2a))}\right].
\end{aligned}$$

Since $\mathrm{e}^{\mu_1 s^2 a} = 1+O(s^2)$, $1<\alpha\leq 2$, and

$$\begin{aligned}
&E_{(0,0)}\left[\mathrm{e}^{-\mu_0\sigma(s^2a)-\mu_1\Xi(\sigma(s^2a))-\mu_2 L_Y^{-1}(\sigma(s^2a))}\right]\\
&= 1-\frac{\exp(I_\Phi(\mu_0,\mu_1,\mu_2))}{\sqrt{\Phi(1,0)}\Gamma\left(1+\frac{\alpha-1}{2}\right)}a^{(\alpha-1)/2}s^{\alpha-1}(1+o(1)),
\end{aligned}$$

we have

$$1-\mathrm{e}^{\mu_1 s^2 a}E_{(0,0)}\left[\mathrm{e}^{-\mu_0\sigma(s^2a)-\mu_1\Xi(\sigma(s^2a))-\mu_2 L_Y^{-1}(\sigma(s^2a))}\right]$$

$$= \frac{\exp(I_\Phi(\mu_0,\mu_1,\mu_2))}{\sqrt{\Phi(1,0)}\Gamma\left(1+\frac{\alpha-1}{2}\right)} a^{(\alpha-1)/2} s^{\alpha-1}(1+o(1)). \quad \square$$

Proof (Corollary 1.2)**.** It is elementary to obtain the statement by the strong Markov property at the instant T_0^Y, Theorem 1.3(i), and a Tauberian theorem. $\square$

4. The case for independent symmetric stable Lévy processes with different indices

Let $1 < \alpha \le 2,\ 0 < \beta \le 2$, and $(X(t), Y(t))$ be such that $X(t)$ and $Y(t)$ are independent, $X(t)$ is symmetric β-stable, and $Y(t)$ is symmetric α-stable. In terms of the characteristic exponent, $\Psi(\xi_1,\xi_2) = |\xi_1|^\beta + |\xi_2|^\alpha$. When $(X(t), Y(t))$ is started from $(x_0, y_0) \in \mathbb{R}^2$, its probability and expectation are denoted by $P_{(x_0,y_0)}$ and $E_{(x_0,y_0)}$, respectively. Let $L_Y(t)$ be the local time at 0 for $Y(\cdot)$: $L_Y(t) = \lim_{\varepsilon\to+0} \frac{1}{2\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(Y(s))\mathrm{d}s$.

For $a \in \mathbb{R}$, we set $\tau(a) = \inf\{t \ge 0 | Y(t) = 0, X(t) \ge a\}$.

We define, for $z \in \mathbb{C}_+$, $\xi_1 \in \mathbb{R}$, and $\mu_i \ge 0$ $(i = 0, 1, 2)$ such that $\mu_0 + \mu_2 > 0$,

$$\Phi_{\alpha,\beta}(\xi_1,\mu_2) = 2\pi \Big/ \int_{\mathbb{R}} \frac{\mathrm{d}\xi_2}{\mu_2 + |\xi_1|^\beta + |\xi_2|^\alpha},$$

$$I_{\alpha,\beta}(\mu_0,\mu_1,\mu_2) = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{2\pi(t^2+1)} \log(\mu_0 + \Phi_{\alpha,\beta}(\mu_1 t,\mu_2)),$$

$$\varphi_{\alpha,\beta}(z;\mu_0,\mu_2) = \exp\left(\frac{-1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{z}{t^2 - z^2}\log(\mu_0 + \Phi_{\alpha,\beta}(t,\mu_2))\mathrm{d}t\right).$$

For $\mu_0 = \mu_2 = 0$, we define $I_{\alpha,\beta}(0,\mu_1,0) = \log\left(\sqrt{C_2(\alpha)}\mu_1^{\beta(\alpha-1)/(2\alpha)}\right)$ and $\varphi_{\alpha,\beta}(z;0,0) = \frac{1}{\sqrt{C_2(\alpha)}}(-\mathrm{i}z)^{-\beta(\alpha-1)/(2\alpha)}$.

We state without proof the following theorem, which we may prove by the same method as in Section 3.

Theorem 4.1. *Let $a > 0$, $\mu_i \ge 0$ $(i = 0, 1, 2)$, and $\mu_0 + \mu_1 + \mu_2 > 0$.*

(i) *It holds*

$$E_{(-a,0)}\left[\mathrm{e}^{-\mu_0 L_Y(\tau(0)) - \mu_1 X(\tau(0)) - \mu_2\tau(0)}\right]$$
$$= \mathrm{e}^{\mu_1 a} - \mathrm{e}^{\mu_1 a}\exp(I_{\alpha,\beta}(\mu_0,\mu_1,\mu_2))\int_{-\infty}^{\infty}\mathrm{d}\theta\,\frac{1-\mathrm{e}^{-\mathrm{i}a(\theta-\mathrm{i}\mu_1)}}{2\pi\mathrm{i}(\theta-\mathrm{i}\mu_1)}\varphi_{\alpha,\beta}(\theta;\mu_0,\mu_2).$$

(ii) *It holds, as $s \to +0$,*

$$1 - E_{(-a,0)}\left[\mathrm{e}^{-\mu_0 s^{2(\alpha-1)} L_Y(\tau(0)) - \mu_1 s^{2\alpha/\beta} X(\tau(0)) - \mu_2 s^{2\alpha}\tau(0)}\right]$$
$$\sim \frac{\exp(I_{\alpha,\beta}(\mu_0,\mu_1,\mu_2)) a^{\beta(\alpha-1)/(2\alpha)}}{\sqrt{C_2(\alpha)}\Gamma\left(1+\frac{\beta(\alpha-1)}{2\alpha}\right)} s^{\alpha-1},$$

where $\sim$ means that the ratio between both sides converges to 1.

(iii) *It holds, as $T \to +\infty$,*

$$P_{(-a,0)}[\tau(0) > T] \sim \frac{a^{\beta(\alpha-1)/(2\alpha)}}{\Gamma\left(1+\frac{\beta(\alpha-1)}{2\alpha}\right)\Gamma\left(1-\frac{\alpha-1}{2\alpha}\right)} T^{-(\alpha-1)/(2\alpha)}.$$

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